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A Numerical Method for Color Uncertainty

Yoshi Ohno
National Institute of Standards and Technology
Gaithersburg, Maryland USA

ABSTRACT

A method for calculating uncertainties of color quantities has been developed using a numerical approach based on the ISO Guide to the Expression of Uncertainty in Measurement. The uncertainties of any color quantities such as chromaticity coordinates, correlated color temperature, and color rendering indices, arising from uncertainties in spectral values of the source or uncertainties in the wavelength scale, can be calculated with this numerical method with no need of deriving partial derivatives analytically. The partial derivatives — thus the sensitivity coefficients — of a color quantity with respect to each spectral value are obtained by simply calculating the color quantity with a small change of each input spectral value. The uncertainty of the color quantity is then obtained as a combined uncertainty for all spectral values. Since this method uses the color calculation program itself to obtain the partial derivatives, there is no need for an analytical process, and the uncertainties can be calculated for any color quantities no matter how complicated the equations and the process may be.

1. Introduction

The uncertainty of measurement is generally evaluated based on the principles described in the ISO Guide to the Expression of Uncertainty in Measurement [1] (abbreviated as GUM). The uncertainties of color quantities such as chromaticity coordinates, correlated color temperature, and color rendering indices have been difficult to deal with due to the complexity of the equations and calculation process. It was not well known until recently how the uncertainties of these color quantities can be derived from the uncertainties of the spectral quantities or tristimulus values from which the color quantities are calculated.

Analytical expressions to calculate the uncertainties of color quantities including tristimulus values, chromaticity coordinates, and correlated color temperature, based on the GUM, were recently presented by Gardner [2, 3]. In his work, partial derivatives of the equations for these color quantities were derived analytically to obtain the sensitivity coefficients with respect to the spectral value. This approach, however, is difficult to apply for quantities such as color rendering indices°[4] and quantities in various color difference formulae [5], in which the calculation process is too complicated to derive the partial derivatives analytically.

To overcome this limitation, an alternate method for calculating uncertainties using a numerical approach based on the GUM has been developed. The partial derivatives — thus the sensitivity coefficients — of a color quantity with respect to each spectral value are obtained by simply calculating the color quantity with a small change of each input spectral value. The uncertainty of the color quantity is then obtained as the combined uncertainty for all spectral values. Since this method uses the color calculation program itself to obtain the partial derivatives, there is no need for analytical process, and the uncertainties can be calculated for any color quantities no matter how complicated the equations and the process may be.

2. Principles

A color quantity q (any color quantity such as chromaticity coordinates or color rendering index R_a) is given as a function of spectral quantity $S(\lambda)$ by

$$q = f\{S(\lambda)\} = f\{S(\lambda_1), S(\lambda_2), ..., S(\lambda_n)\}. \quad (1)$$

The partial derivative of q with respect to $S(\lambda_i)$ is numerically obtained at each wavelength by

$$\frac{\partial q}{\partial S(\lambda_i)} = \frac{1}{2\Delta S}\left[f\{S(\lambda_1), S(\lambda_2), ..., S(\lambda_i) + \Delta S, ...S(\lambda_n)\} - f\{S(\lambda_1), S(\lambda_2), ..., S(\lambda_i) - \Delta S, ...S(\lambda_n)\}\right]; \quad i = 1, ..., n \quad (2)$$

where ΔS is chosen to be small enough relative to the average value of $S(\lambda)$. In practical applications, this can be simplified to:

$$\frac{\partial q}{\partial S(\lambda_i)} = \frac{1}{\Delta S}\left[f\{S(\lambda_1), S(\lambda_2), ..., S(\lambda_i)+\Delta S, ...S(\lambda_n)\} - f\{S(\lambda)\}\right]; \quad i = 1, ..., n \qquad (3)$$

This partial derivative is called the sensitivity coefficient (of q with respect to each spectral value). Assuming that the input quantities (the spectral values) are not correlated, the combined standard uncertainty $u(q)$ of the color quantity is given from the standard uncertainties $u\{S(\lambda_i)\}$ of the spectral values by

$$u(q) = \sqrt{\sum_{i=1}^{n}\left\{\frac{\partial q}{\partial S(\lambda_i)}\right\}^2 u^2\{S(\lambda_i)\}}. \qquad (4)$$

Care must be taken to ensure that the value for ΔS is chosen appropriately. If ΔS is too large, the result will deviate from the true differential value. If ΔS is too small, the resulting changes in the color quantity will be too small for the precision of the computer calculation, leading to large errors. This can be verified by repeating the calculation with ΔS changed by one or two orders of magnitude to check whether the same results are obtained. A test using a commercial spreadsheet application with double precision (16 digit) calculation showed that ΔS in the range of 10^{-3} to 10^{-10} of the maximum value of the spectra produced the same results (to 6 digits) of (x,y) uncertainties for the spectra of various fluorescent lamps, LEDs, and the color samples used in Ref. [4].

3. Verification

For verification, the standard uncertainties of the (u, v) chromaticity coordinates and correlated color temperature of the four light sources (Fig. 1) used in Gardner s analysis [3] have been calculated with this numerical method under the same conditions. The relative standard uncertainty of the spectral value, a, is set to 0.01, constant over the entire spectral region. The results of the calculation with the numerical method are shown in Tables 1 and the differences from the Gardner s results are shown in Table 2. As shown in Table 2, the results obtained by the numerical method agreed with Gardner s with negligible differences. Further, the numerical method successfully calculated the uncertainty of the general color rendering index R_a.

Figure 2 shows the sensitivity coefficients of chromaticity coordinates x, y with respect to $S(\lambda)$ for sources S2 and S7, two dissimilar spectral distributions. The results indicate that the sensitivity coefficient curves are fairly similar in shape even for sources with dissimilar spectral distributions and

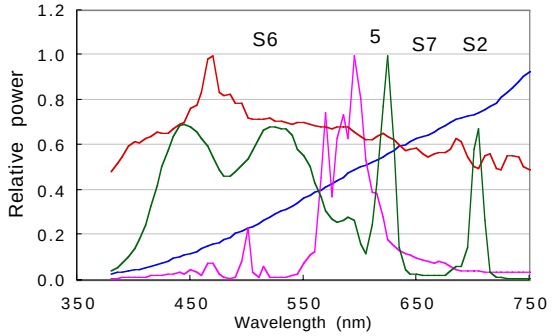

Figure 1. Spectral power distributions of the four light sources analyzed.

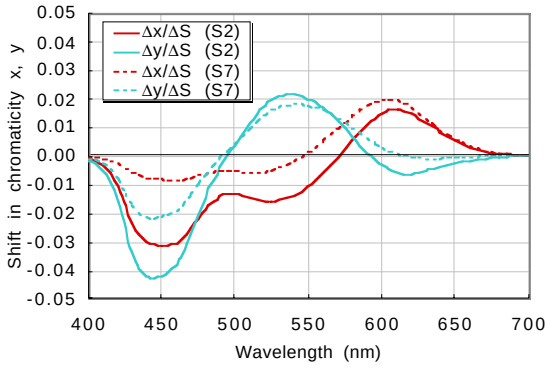

Figure 2. Sensitivity coefficients of chromaticity x, y with respect to $S(\lambda)$ for sources S2 and S7.

Table 1. Standard uncertainties of chromaticity (u, v) for sources S2 — S7, calculated with the numerical method.

Source		S2	S5	S6	S7
u		0.2556	0.3015	0.2014	0.1736
v		0.3506	0.3599	0.3106	0.3048
T_c[K]		2856	2043	6415	9314
$u_c(u)$	a= 0.01	0.0003	0.0002	0.0002	0.0002
$u_c(v)$		0.0001	0.0000	0.0002	0.0002
$u_c(T_c)$ [K]		5.9	3.0	19.2	41.8
$u_c(R_a)$		0.17	0.14	0.17	0.12

Table 2. Comparison with Gardner's result.

		Difference (Numerical method - Ref.[3])			
Source		S2	S5	S6	S7
Δu		0.0000	0.0000	0.0000	0.0000
Δv		0.0000	0.0000	0.0000	0.0000
ΔT_c[K]		0	0	-1	4
$\Delta u_c(u)$	a= 0.01	0.0000	0.0000	0.0000	0.0000
$\Delta u_c(v)$		0.0000	0.0000	0.0000	0.0000
$\Delta u_c(T_c)$ [K]		-0.1	-0.1	0	-1

that the curves have peaks at wavelengths roughly corresponding to the peaks of the CIE color matching functions. The data of colored light sources indicate that the sensitivity is higher in the regions where there are little or no emissions.

4. Color uncertainty arising from wavelength uncertainties

In the uncertainty analysis mentioned above, the uncertainty $u(S_i)$ of the spectral data is normally obtained from the uncertainty of the spectral irradiance standard lamp and the uncertainty of transfer to the test light source at each wavelength. However, there are also uncertainties in the wavelength scale of spectroradiometers, which also affect the measured color quantities. Such effects of wavelength uncertainties tend to be insignificant for broadband light sources such as incandescent lamps, but can be critical for light sources containing narrowband emissions — such as tri-phosphor fluorescent lamps, light emitting diodes, and color displays, especially liquid crystal displays (LCDs). To calculate the uncertainty of color quantities arising from wavelength uncertainties $u(\lambda_i)$, the partial derivative of a color quantity q with respect to wavelength λ_i is numerically obtained at each wavelength by

$$\frac{\partial q}{\partial \lambda_i} = \frac{1}{\Delta\lambda}\left[f\{S(\lambda_1), S(\lambda_2), ..., S(\lambda_i + \Delta\lambda), ... S(\lambda_n)\} - f\{S(\lambda)\}\right] \quad ; \; i = 1, ..., n \qquad (5)$$

where $\Delta\lambda$ is chosen to be small enough (e.g., 0.1˚nm) relative to the measurement interval of $S(\lambda)$. $S(\lambda_i + \Delta\lambda)$ can be obtained by appropriate interpolation such as Lagrange interpolation. Assuming that the input quantities (wavelength errors) are not correlated, the combined standard uncertainty $u(q)$ of the color quantity is given from the standard uncertainties $u(\lambda_i)$ of the wavelength scale by

$$u(q) = \sqrt{\sum_{i=1}^{n} \left\{ \frac{\partial q}{\partial \lambda_i} \right\}^2 u^2(\lambda_i)} \qquad (6)$$

For example, the uncertainties in chromaticity *x, y* of the LCD spectra as shown in Fig. 3 were calculated using Eqs. (5) and (6). The spectral data at 5 nm intervals were first shifted by 0.1 nm by Lagrange interpolation in order to allow calculations of Eq. (5). The partial derivatives (sensitivity coefficients) obtained by Eq. (5) for the white LCD spectrum are shown in Fig. 4. The results in this example indicate that sharp slopes in spectra cause large uncertainties in chromaticity due to

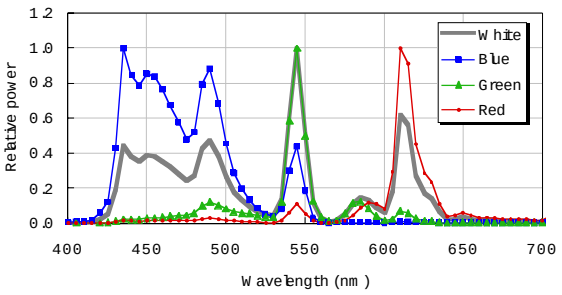

Figure 3. LCD spectra measured at 5 nm intervals with 5 nm bandpass.

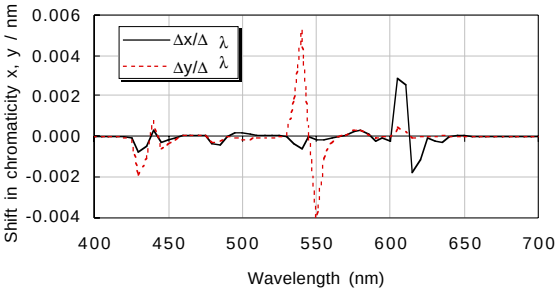

Figure 4. Sensitivity coefficients of chromaticity coordinates *x, y* with respect to wavelength λ_i for the white LCD spectrum in Fig.3.

Table 3. Standard uncertainties of the chromaticity coordinates (*x, y*) for the LCD spectra shown in Fig. 4, due to wavelength uncertainty of 0.1 nm.

	White	Blue	Green	Red	S2
$u_c(x)$	0.0005	0.0001	0.0003	0.0004	0.0000
$u_c(y)$	0.0008	0.0005	0.0008	0.0003	0.0000
$u_c(T_c)$ [K]	54				0.41
T_c[K]	9314				2856

wavelength uncertainties. Table 3 shows the results of the calculation using Eq. (6) - the standard uncertainties of chromaticity *x,* ˚*y* for this set of LCD spectra, as well as for source S2 in Fig. 1 (tungsten lamp sphere source), where $u(\lambda_i)$=0.1˚nm is assumed for all the wavelengths. The results demonstrate dramatic differences in the uncertainties between LCD spectra and a tungsten source. These and other data indicate that the uncertainties of chromaticity of light sources arising from wavelength uncertainties depend not only on the slope but also on the width of the peaks in the spectral distribution curve.

Note that the wavelength errors in real spectroradiometers are more or less correlated

between neighboring wavelengths. The uncertainty values calculated using Eq. (6), which ignores the correlation, tend to be overestimated, particularly for sources containing narrow-band peaks.

5. Conclusions

With this numerical method, the uncertainties of color quantities such as chromaticity coordinates, correlated color temperature, and even color rendering indices associated with uncertainties in spectral values or in wavelength scale can be calculated directly, eliminating the need to derive partial derivatives analytically. This method can be applied to the uncertainties of any other color quantities such as the CIE $L^*a^*b^*$ coordinates and metamerism indices utilizing the computer program to calculate these quantities.

Further, the method will also be useful for calculating the uncertainties of complicated photometric quantities such as the value of f_1' and spectral mismatch correction factors of a photometer from the uncertainties of its spectral responsivity, and uncertainties of any other quantities calculated from spectral values.

The combined uncertainty described in this paper assumes no correlations between input spectral data. Correlations can significantly affect the combined uncertainty and need to be studied to develop appropriate treatment.

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